

Automatic Calculation of 2-loop Weak Corrections to Muon Anomalous Magnetic Moment *

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An automatic system to calculate two loop weak corrections to muon anomalous magnetic moment is discussed. Diagrams are classified into eight types, according to their topology. Adopting Civitanović-Kinoshita representation of Feynman amplitude and using the topological property, the renormalization is performed consistently by n -dimensional regularization method.

1. Introduction

GRACE system¹ has been extended to produce one loop Feynman amplitude in automatic way and the order α corrections to several $2 \rightarrow 2$ reactions are calculated². The GRACE can also generate higher loop diagrams and corresponding amplitudes. As a first application of the system to two-loop calculation, we consider the weak corrections to muon anomalous magnetic moment.

QED corrections to muon ($g-2$) are calculated up to eighth order and its value is³

$$\Delta a_{QED}^{(8)} = (g-2)/2 = 1165846947(\pm 46 \pm 28) \times 10^{-12}$$

The first error arises from theoretical uncertainty and the second one from a measurement uncertainty of the fine structure constant α . The next order corrections in QED are estimated as $\Delta a_{QED}^{(10)} = (39 \pm 10) \times 10^{-11}$. The vacuum polarization due to hadrons contributes $\Delta a_{hadron} = (703 \pm 19) \times 10^{-10}$. The weak interaction calculated at one loop level contribute $\Delta a_{Weak}^{1-loop} = (195 \pm 1) \times 10^{-11}$. The two loop weak correction is estimated in the leading logarithmic approximation and it reaches about 22 % of the one loop result⁴. So, it is worth to get two-loop contributions from 1678 diagrams completely in automatic way.

2. Formalism

We adopt the Civitanović-Kinoshita⁵ representation of the Feynman amplitude.

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Combining the denominators by Feynman parameters, the general muon vertex at two loop level (in the case of 6 internal lines) is expressed as

$$\Gamma_\mu = \Gamma(6) \int \prod dz_j \delta(1 - \sum_j z_j) \int \frac{d^n \ell_1}{(2\pi)^n} \frac{d^n \ell_2}{(2\pi)^n} \frac{F(D)}{\sum_j z_j (p_j^2 - m_j^2)} \quad (1)$$

where, p_j is a momentum flowing on the internal line j and expressed as

$$p_j = \sum_{s=1,2} \eta_s(j) \ell_s + q_j. \quad (2)$$

The symbol $\eta_s(j)$ represents (1,-1,0) according to the flow of loop momentum ℓ_s and q_j is a external momentum flow on the line j . The $F(D)$ represents a numerator and is expressed by several momenta and γ matrices. By shifting the loop momenta to delete linear terms, the denominator becomes

$$\text{Denom.} = \sum_{s,t} U_{st} (\ell_s \cdot \ell_t) - V \quad (3)$$

$$V = \sum_j z_j m_j^2 - \sum_j z_j (q_j \cdot q_j) + \frac{1}{\det U} \sum_{i,j} z_i z_j B_{ij} (q_i \cdot q_j), \quad (4)$$

where

$$U_{s,t} = \sum_{j=1}^6 z_j \eta_s(j) \eta_t(j) \quad (5)$$

and

$$B_{ij} = \sum_{s,t} \eta_s(i) \eta_t(j) U_{st}^{-1} \cdot \det U = B_{ji}. \quad (6)$$

These matrices depend only on the topology of the diagram. By changing the basis of the independent loop momenta, the matrix U is diagonalized and we rescale the loop momenta. We adopt n -dimensional regularization method to remove the Ultra Violet divergence. As the numerator can be reproduced by the differential-integral operator D_j^μ , we first integrate over n -dimensional loop momenta. The operator D_j^μ (j represents the line number) is defined as follows.

$$D_j^\mu = \frac{1}{2} \int_{m_j^2}^{\infty} dm_j^2 \frac{\partial}{\partial q_{j\mu}} \quad (7)$$

The basic relation to pick up the numerator is

$$D_j^\mu \frac{1}{(p_j^2 - m_j^2)} = \frac{p_j^\mu}{(p_j^2 - m_j^2)}. \quad (8)$$

The fundamental relations necessary for us are given by

$$D_j^\mu \frac{1}{V^m} = \frac{Q_j'^\mu}{V^m} \quad (9)$$

$$D_i^\mu D_j^\nu \frac{1}{V^m} = \frac{Q_i'^\mu Q_j'^\nu}{V^m} + \left(-\frac{1}{2 \det U} \right) \frac{g^{\mu\nu}}{(m-1)} \frac{B_{ij}'}{V^{m-1}} \quad (10)$$

where

$$Q_j'^\mu = q_j^\mu - \frac{1}{\det U} \sum_i z_i B_{ij} q_i^\mu = -\frac{1}{\det U} \sum_i z_i B_{ij}' q_i^\mu \quad (11)$$

The numerator can be reproduced by replacing p_j^μ with the operator D_j^μ . The basic formula for our calculation becomes (except for coupling constant),

$$\begin{aligned} \Gamma^\mu = & \frac{1}{(4\pi)^n} \int \prod dz_j \delta(1 - \sum_j z_j) \left[\frac{\Gamma(6-n)}{(\det U)^{n/2}} \frac{F_0^\mu}{(V - i\varepsilon)^{6-n}} \right. \\ & \left. + \frac{\Gamma(5-n)}{-2 (\det U)^{n/2+1}} \frac{F_1^\mu}{(V - i\varepsilon)^{5-n}} + \frac{\Gamma(4-n)}{4 (\det U)^{n/2+2}} \frac{F_2^\mu}{(V - i\varepsilon)^{4-n}} \right] \quad (12) \end{aligned}$$

Practical method to generate proper numerator is as follow.

- replace p_j^μ in numerator $\rightarrow (ck_j^\mu + Q_j'^\mu)$
- c^0 terms $\rightarrow F_0^\mu$
- c^2 terms $\rightarrow F_1^\mu$: replace $k_i^\mu k_j^\nu \rightarrow (-1/(2 \det U)) B_{ij}' g^{\mu\nu}$
- c^4 terms $\rightarrow F_2^\mu$: replace $k_i^\mu k_j^\nu k_k^\lambda k_\ell^\sigma \rightarrow (-1/(2 \det U))^2 \left\{ g_{\mu\nu} g_{\lambda\sigma} B_{ij}' B_{k\ell}' + g_{\mu\lambda} g_{\nu\sigma} B_{ik}' B_{j\ell}' + g_{\mu\sigma} g_{\nu\lambda} B_{i\ell}' B_{jk}' \right\}$

In the above replacement, the contraction of Lorentz indices should be done in n -dimension. We introduce the following projection operator in 4-dimension to pick up the contribution to $(g-2)$. (q :photon mom., $(p-q/2)$, $(p+q/2)$:muon mom.)

$$\text{Proj}(\mu) = \frac{1}{4} (\not{p} - \frac{1}{2} \not{q} + m) (m \gamma_\mu(p.p) - (m^2 + \frac{q.q}{2}) p_\mu) (\not{p} + \frac{1}{2} \not{q} + m) \quad (13)$$

By this projection, the F_2 term drops out. We replace F_0, F_1 with the projected one f_0, f_1 . The factor $(-1/2)$ is included in f_1 .

3. Removal of the Ultra Violet Divergence

The ultra violet (UV) divergence arises from the one loop sub diagram⁶. The over all divergence (F_2 term) is irrerevant in our case, because of the projection. It is necessary for order (α^2) charge renormalization. The diagrams are classified into 8 different types of topology. They are shown in the Fig.1. We will discuss two types of topology as examples. The vertex and self-energy type.(See Fig.2.)

3.1. Vertex type

We divide the diagram into a sub diagram (S) and the remained (R). The corresponding Feynman parameters are

$$(z_2, z_3, z_6) \in S, \quad (z_1, z_4, z_5) \in R \quad (14)$$

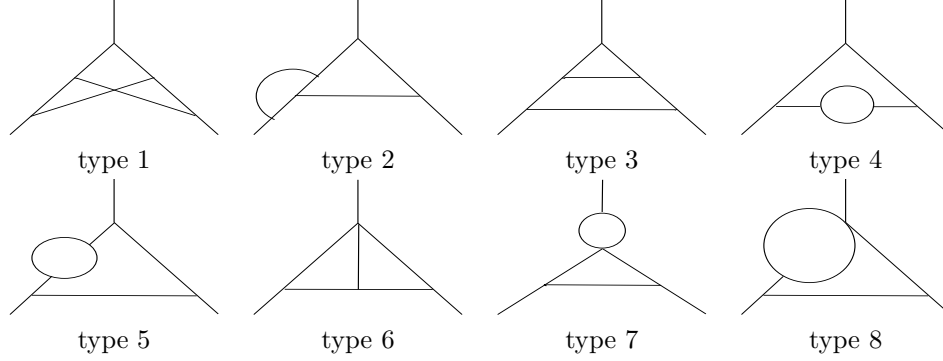


Figure 1: Types of Topology

The determinant ($\det U$) has a semi-factorized form in the parameters. This implies that the singularity occurs when all the parameters in sub-diagram S tends to 0. In order to see this clearly, we reparametrize the Feynman parameters as follows.

$$z_2 = x(1 - y), \quad z_3 = xy(1 - z), \quad z_6 = xyz \quad (15)$$

$$z_1 = (1 - x)(1 - u), \quad z_4 = (1 - x)u(1 - v), \quad z_5 = (1 - x)uv \quad (16)$$

The measure of integration is changed into $x^2(1 - x)^2yu$. When x approaches 0 then the $\det U$ also tends to 0. This is the origin of divergence. It is important that the structure of determinant only depends on the topology of the diagram and the type of different topology is restricted in small number. So we can prepare the subtraction procedures beforehand, according to the type of topology. The UV divergence occurs from the second term.

- the second term $f1$

Setting $\varepsilon = 2 - n/2$, the main part of the integrand becomes

$$\Gamma(1 + 2\varepsilon) \frac{x^2(1 - x)^2yu}{x^{3-\varepsilon}u(x, y, z)^3} \frac{f10 + \varepsilon \cdot f11}{V^{1+2\varepsilon}} (4\pi)^{2\varepsilon} u(x, y, z)^\varepsilon \quad (17)$$

where $\det U = x \cdot u(x, y, z)$. (We must keep numerators up to $O(\varepsilon)$.) We can see the existence of the singularity at $\varepsilon = 0$ when $x \rightarrow 0$, due to the factor $x^{\varepsilon-1}$. In order to extract the singularity, let us consider the following integral.

$$I = \int_0^1 x^{\varepsilon-1} (h_0(x) + \varepsilon \cdot h_1(x)) dx \quad (18)$$

Partial integration and the expansion in ε give

$$I = \frac{1}{\varepsilon} h_0(0) + h_1(0) - \int_0^1 \log x \frac{dh_0(x)}{dx} \quad (19)$$

Using the above formula, the contribution from $f1$ term turns out (as the integrand of $dydzdudv\{yu\}$)

$$F_2^{(4)}(0) = \left(\frac{1}{\varepsilon} + 2\log(4\pi) - 2\gamma_E \right) \frac{f10(0)}{V(0)} + \frac{f10(0)}{V(0)}(-2\log V(0)) + \frac{f11(0)}{V(0)} - \int_0^1 \log(x) \frac{\partial}{\partial x} \left(\frac{(1-x)^2}{u(x,y,z)^3} \frac{f10(x)}{V(x)} \right) dx \quad (20)$$

in unit of $(\alpha/\pi)^2/16$.

3.2. Self energy type

Similarly, we divide the diagram into a sub diagram (S) and the remained (R). The corresponding Feynman parameters are

$$(z_2, z_5) \in S, \quad (z_1, z_3, z_4, z_6) \in R \quad (21)$$

Reparametrization of Feynman parameters is done in the similar way.

$$z_2 = x(1-y), \quad z_5 = xy \quad (22)$$

$$z_1 = (1-x)(1-u), \quad z_3 = (1-x)u(1-v), \quad z_4 = (1-x)uv(1-w), \quad z_5 = (1-x)uvw \quad (23)$$

The measure is slightly different from the previous case, $x(1-x)^3u^2v$. The singularity occurs even in the first term $f0$. The treatment is the same as the $f1$ in the previous case. The second term has the different structure. The main part of the integrand is

$$\Gamma(1+2\varepsilon) \frac{x(1-x)^3u^2v}{x^{3-\varepsilon}u(x,y)^3} \frac{f10(x) + \varepsilon \cdot f11(x)}{V^{1+2\varepsilon}} (4\pi)^{2\varepsilon} u(x,y)^\varepsilon. \quad (24)$$

To discuss the singularity, let us consider the following integral.

$$I = \int_0^1 x^{\varepsilon-2} (g_0(x) + \varepsilon \cdot g_1(x)) dx \quad (25)$$

Partial integration and the expansion in ε give

$$I = \frac{1}{\varepsilon} \frac{dg_0(0)}{dx} + \frac{dg_0(0)}{dx} + \frac{dg_1(0)}{dx} - g_0(1) - \int_0^1 \log x \frac{d^2g_0(x)}{dx^2} dx \quad (26)$$

By this formula, we can extract UV divergent part and the finite contribution.

4. Counter terms

There are 58 diagrams including one loop counter term. The contributions from these diagrams are easily obtained, by interpreting the counter term as the new coupling including $(1/\varepsilon)$. No divergence occurs in one loop integration, however, we must keep order ε quantity in the integration to get final result.

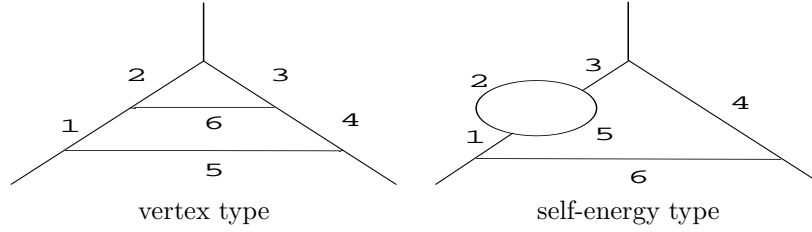


Figure 2: Assignment of Feynman Parameters

5. System of Calculation

We first prepare the topology dependent part. Corresponding to each topology, we can assign the flow of loop momenta and external momenta. Then we can calculate the important quantity $\det U$, B_{ij} and the coefficients expressing the weight of the flow of momenta q and p in each internal line by using the conservation law of momentum at each vertex. These are the prepared files. The GRACE generates diagrams and corresponding FORM source codes. By invoking FORM, the prepared files are included according to their topology and FORTRAN source codes giving the UV and finite part contribution to $(g - 2)$ are generated.

Numerical integration over Feynman parameter space is done by BASES⁷. For test run, we have checked several diagrams. We have reproduced the pure QED results by this system. The UV divergence is extracted systematically. The infrared (IR) singularity is regulated by introducing a tiny photon mass. The diagrams including self-energy type sub-diagram develop superficial IR singularity due to the fact that the x -derivative to get rid of UV part increases the power of denominator function. We need subtle treatment for IR singularity.

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